Exercises for lecture 2

Exercise 1: Maxwell theory in differential form notation

In this exercise we assume a flat Lorentzian metric, $g_{\mu\nu} = \eta_{\mu\nu}$. Recall that the current one-form can be written in components as $J = J_\mu dx^\mu = \rho dx^0 + j_i dx^i$.

a. Show that the inhomogeneous Maxwell equations of motion, $\nabla \cdot E = \rho$ and $\nabla \times B - \partial E/\partial t = j$ can be written as $d \star F = \star J$.

b. What does the equation $d^2 = 0$ imply for ρ and j? Explain that the resulting equation can be interpreted as conservation of charge.

c. To gauge fix the Maxwell gauge symmetry, one often chooses the condition $\partial_\mu A^\mu = 0$. Write this condition in differential form notation.

Exercise 2: Equations of motion for Maxwell theory

Show that the Euler-Lagrange equations for the action $S = \int F \wedge \star F + A \wedge \star J$ are indeed Maxwell's homogeneous equations of motion (2.49).

* Exercise 3: The Dirac monopole (hand-in exercise)

In this exercise, we will work with three-dimensional polar coordinates (r, θ, ϕ), defined by

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

in terms of the cartesian coordinates (x, y, z). Of course, this map is not 1-to-1; in particular, θ and ϕ are only defined up to multiples of 2π. The time coordinate t will not play a role in this exercise, so you may assume we are working on $\mathbb{R}^3$.

We will consider the field strength

$$F = \frac{g}{4\pi} \sin\theta \, d\theta \wedge d\phi$$

with $g \neq 0$.

a. Compute $\int_{S^2} F$, where F is a two-sphere centered at the origin of $\mathbb{R}^3$.

b. Use the previous result and Stokes' theorem to show that F can not be a closed form.

c. Naively computing dF, one still seems to find $dF = 0$. How can this apparent contradiction with the result of (b) be understood?

d. One way to understand the result of (c) is to compute $\star F$. Show that indeed $\star F$ has a singularity at the origin of $\mathbb{R}^3$.

Summarizing, we have found that our field strength F is well-defined and closed only on $(\mathbb{R}^3)^* \equiv \mathbb{R}^3 \backslash \{0,0,0\}$. As this manifold is not topologically trivial, we cannot use Poincaré's lemma to conclude that $F = dA$ everywhere. However, if we define D^+ and D^- as the regions where $\theta \neq \pi$ and $\theta \neq 0$ respectively (that is, D^+ is $\mathbb{R}^3$ excluding the negative z-axis, and D^- is $\mathbb{R}^3$ excluding the positive z-axis), these two regions *are* topologically trivial. On these regions, we now consider the 1-forms

$$A^+ = \frac{g}{4\pi}(1 - \cos\theta)d\phi, \qquad A^- = -\frac{g}{4\pi}(1 + \cos\theta)d\phi$$

e. (Easy:) Show that $F = dA^+$ and $F = dA^-$ in D^+ and D^- respectively.

f. Compute $A^+ - A^-$. Where is this 1-form defined? In particular: is that space topologically trivial? Can A^+ be obtained from A^- using a gauge transformation?

One can slightly generalize the concept of a gauge transformation as follows. In quantum mechanics, one is interested in the wave function $\psi(x)$ of a particle. Under a transformation of the potential

$$A \to A + \omega$$

with ω a closed one-form, the wave function transforms as

$$\psi(x) \to \psi(x) \exp\left(i \int_\gamma \omega\right)$$

where γ is a path from an arbitrarily chosen base point to the point to x. A *large gauge transformation* is a transformation of A by a one-form ω such that the transformation of $\psi(x)$ is well-defined.

g. In the previous sentence, "well-defined" means that the transformed value of $\psi(x)$ should not depend on the choice of a path γ. Argue that in a topologically trivial space, this condition is automatically satisfied if ω is closed.

h. Which additional condition should ω satisfy if the space is not topologically trivial? (In particular: if it is not simply connected?)

i. In our example, which condition should g satisfy so that A^+ and A^- are related by a large gauge transformation?

The upshot of this exercise is therefore that, assuming the condition found in (i) is satisfied, our field strength F describes a "good" electromagnetic field configuration on $\mathbb{R}^3 \backslash \{0,0,0\}$. The interpretation of this configuration is that there is a "defect" at the singular point in the origin – an object which can be interpreted as a particle.

j. Show that this particle does not have an electric charge, but that by replacing the E-field with the B-field, it can be considered to have a "magnetic charge". This particle (which has never been observed in nature!) is called the Dirac magnetic monopole.